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A CLASS OF EXACT SOLUTIONS FOR ANISOTROPIC MAGNETOFLUID SPACE-TIME ADMITTING CONFORMAL MOTIONS

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Abstract: *The implications of a space-time admitting one parameter group of conformal motions on the dynamical variables involved in the self-gravitating Magnetofluid which is anisotropic in nature are examined. Moreover, it is proved that the symmetry vector chosen is different direction generates two kinds of equations of state. It is shown that the magnetic field vector is normal to the plane of rotation if it is one of the symmetry vectors.*

Keywords: *Conformal Motions, Anisotropic Magnetofluid.*

1. Introduction:

The study of geometrical symmetries like motions and collineations is motivated by the necessity of discovering conservation laws and to throw light on unknown properties of dynamical variables involved in the study of relativistic distributions of matter. One of the symmetries known as conformal motion as described by Davis[1] has the defining expression

$$L_x g_{ab} = \psi g_{ab}$$

Here, L_x is the Lie derivative with respect to vector field $\bar{x}$ *and* ψ is an arbitrary function of co-ordinates. It is found by Herrera et al [3] that under this symmetry of conformal motions the Einstein's field equations corresponding to anisotropic matter yields a class of spherically symmetric and static solutions. By tracing out the similarity between the stress energy tensor for anisotropic matter distribution as given by Herrera et al [3] and the stress energy tensor for Magnetofluid as given by A. Lichnerowicz [4], we have suggested stress energy tensor for Magnetofluid which is anisotropic in character.

The aim of this paper is to study the dynamical aspect of this anisotropic Magnetofluid when the space time admits the geometrical symmetries called as conformal motion in context of the general theory of relativity. Less weightage is given to the properties of anisotropy of a fluid.

In section 2, the stress energy tensor for Magnetofluid with anisotropic nature is designed and in section 3, the necessary field equations are stated. The equations of motions of the Magnetofluid are derived in section 4. In section 5, we discuss the effects of conformal motions on dynamical properties of the Magnetofluid. The corresponding conservation law generator is derived. Throughout the investigations, we have considered four dimensional Lorentzian manifold with signature (-,-,-, +)

2. Anisotropic Magnetofluid

The stress energy tensor characterizing the relativistic anisotropic fluid is given by Herrera et al [3]

$$T_{ab} = (\rho + P_1)U_a U_b - P_1 g_{ab} + (P - P_1)x_a x_b \; \ldots \; (2.1)$$

Where U_a is the four velocity of the fluid, x_a is unit space like vector orthogonal to U_a, ρ is the energy density, P is the pressure in the direction of x_a and P_1 is the pressure in the direction perpendicular to U_a and x_a both (i.e. along S_a, $S_a U^a = S_a H^a = 0$).

We have the stress energy tensor for relativistic charged perfect fluid with infinite conductivity and constant magnetic permeability formulated by A. Lichnerowicz [4] in the form

$$T_{ab} = (\rho + P + \mu h^2)U_a U_b - (P + ½\mu h^2)g_{ab} - \mu h_a h_b. \quad \ldots (2.2)$$

Here, ρ is the matter energy density, P is the isotropic pressure, μ is the magnetic permeability which is constant. The time like vector U_a and space like vector h_a satisfy the properties:

$$U_a U^a = 1, \; h_a h^a = -h^2, U_a h^a = 0. \quad \ldots (2.3)$$

By readjusting the terms in (2.2) we can write this expression as

$$T_{ab} = (\bar{\rho} + \bar{P})U_a U_b - \bar{P} g_{ab} + (P - \bar{P})H_a H_b. \quad \ldots (2.4)$$

The term involved here have the meanings.

$\bar{\rho}(= \rho + ½\mu h^2)$: Matter energy density for the Magnetofluid.

$\bar{P}(= P + ½\mu h^2)$: The pressure in the direction normal to $U_a$andH_a.

$H_a(= h^{-1}h_a)$: The unit space like magnetic field vector.

$P(= P - ½\mu h^2)$: Pressure in the direction ofH_a.

It follows from (2.1) and (2.4) that the stress energy momentum tensor given by (2.4) also represents an anisotropic character. The form (2.4) generates the stress energy tensor for isotropic perfect fluid when

$$P = \bar{P} \Rightarrow \mu h^2 = 0 \quad \ldots (2.5)$$

Thus we have after refer to stress energy tensor given by (2.4) for the anisotropic Magnetofluid. The anisotropy seems to be generated due to magnetic field.

Note: We get from (2.4)

$$T_{ab}U^a U^b = \bar{\rho} = \rho + ½\mu h^2 , \quad \ldots (2.6)$$

$$T_{ab}H^a H^b = P , \quad \ldots (2.7)$$

$$T = T_{ab}g^{ab} = \bar{\rho} - 2\bar{P} - P. \quad \ldots (2.8)$$

Hence it follows from (2.6) that the energy density for the anisotropic Magnetofluid is the time like eigen value is given by (2.7) and the rest mass is given by (2.8).

Remark: The magnetic field involved in (2.4) is subject to satisfy the Maxwell field equations which are given in section3.

3. Field equations

To study the space time of anisotropic Magnetofluid we mainly use the following field equations.

A) Field equations of gravitations

The general relativistic structure of the space time is governed by the well-known Einstein's field equation in the form

$$R_{ab} - \frac{1}{2}R g_{ab} = -KT_{ab}, \quad \ldots (3.1)$$

Where R_{ab} is the Ricci tensor, R is the Ricci scalar, k is the coupling constant and T_{ab} is the stress energy tensor described by (2.4). It follows from (3.1) that

$$R = R^a_a = KT. \quad \ldots (3.2)$$

$$T_{ab} = -K[(\rho + P)U_a U_a - \bar{P} g_{ab} + (P - \bar{P})H_a H_b - \frac{1}{2}T g_{ab} \quad \ldots (3.3)$$

B) Maxwell's field equation

For infinitely conducting charged fluid the only valid set of Maxwell's equations as suggested by A. Lichnerowicz [4] are

$$\{h(U^a H^b - U^b H^a)\}; b = 0. \quad \ldots (3.4)$$

i.e. $U^a H^b h_{;b} - h_{;b} U^b H^a + hU^a H^b{}_{;b} + hH^b U^a{}_{;b} - hU^b H^a{}_{;b} = 0$ ... (3.5)

If we contract this equation with U_a and H_a respectively we get

$H^b h_{;b} + hH^b{}_{;b} - hU_a U^b H^a{}_{;b} = 0$ ... (3.6)

$h_{;b} U^b + hU^b{}_{;b} + hU_{a;b} H^a H^b = 0.$ ... (3.7)

We have the expression for the gradient of the velocity field in terms of Kinematical parameters [2]

$U_a : b = \sigma_{ab} + W_{ab} + \frac{1}{3}\theta h_{ab} + U_a U_b.$ ... (3.8)

Where $h_{ab} = g_{ab} - U_a U_b$.

This when used in(3.6) and (3.7) we get

$h_{;b} H^b + hH^b{}_{;b} + hU^a H_a = 0.$ ... (3.9)

$h_{;b} U^b + \frac{2}{3} h\theta + h\sigma_{ab} H^a H^b = 0.$ ... (4.1)

4. Local conservation Laws

The contracted Bianchi identities provide the local conservation laws by utilizing the Einstein's Field equation. These laws are described as

$T^{ab}{}_{;b} = 0$ ... (4.2)

This equation with the stress energy tensor for Magnetofluid given by (2.4) provides

a) The equation of continuity in the form

$\dot{\bar{\rho}} + (\bar{\rho} + \bar{P})\theta + (P - \bar{P}) H^a{}_{;b} H^b U_a = 0$ ... (4.3)

Where overhead dot denotes the covariant derivative along the flow vector U^a. If we use the Maxwell's equation (3.5) and (4.3) then we get the equation of continuity for the Magnetofluid in the form

$(\dot{\bar{\rho}} - ½\mu h^2) + (\bar{\rho} + P)\theta = 0$ ... (4.4)

b) This equation of continuity when substituted in the general expression of (4.2) we get the equation of streamlines, viz.

$$(\bar{\rho} + \bar{P})\dot{U}^a + \dot{\bar{P}} U^a + (\bar{P} - P) H^a{}_{;b} H^b$$
$$-P_{;b} g^{ab} + (P - \bar{P})_{;b} H^a H^b + (P - \bar{P}) H^a{}_{;b} H^b + (P - \bar{P}) H^a H^b{}_{;b} = 0 \quad \dots (4.5)$$

This equation describes the deviation of fluid path from the geodesic path $\dot{U}^a = 0$

c) The equation (4.2) when contracted with U_a we get

$(\bar{\rho} + \bar{P})\dot{U}^a H_a - P_{;b} H^b - (P - \bar{P}) H^b{}_{;b} = 0$ ... (4.6)

5. Group of conformal motions

A space time is said to admit a group of conformal motions in the direction of arbitrary vector field x_a if the following condition is satisfied [3]

$L_x g_{ab} = \psi g_{ab}$ ... (5.1)

Here, ψ is an arbitrary function of co-ordinates.

Remark: If $\psi = 0$ then (5.1) degenerates into the group of motions. We now study the properties of anisotropic Magnetofluid under the symmetry of group of conformal motions given by (5.1).

Theorem (5.1): For the anisotropic Magnetofluid admitting a group of conformal motions along an arbitrary vector field $\bar{x}$. The following results are equivalent:

(1) $\psi_{;b} = 0$

(2) (a) $L_x \bar{\rho} + \psi \bar{\rho} = 0$

(b) $L_x \bar{P} + \psi \bar{P} = 0$

(c) $L_x P + \psi P = 0$

Proof: The conditions (5.1) provides the results

$$L_xU_a = \left(\frac{\psi}{2}\right)U_a, L_xU^a = -\left(\frac{\psi}{2}\right)U^a. \quad \ldots (5.2)$$

$$L_xH_a = \left(\frac{\psi}{2}\right)H_a. \quad \ldots (5.3)$$

$$L_xH^a = -\left(\frac{\psi}{2}\right)H^a. \quad \ldots (5.4)$$

Now, we find the expression for L_xT_{ab} by making use of the equations (2.4) and (5.1) - (5.4)

$$L_xT_{ab} = [L_x\bar{\rho} + L_x\bar{P} + \psi(\bar{\rho} + \bar{P})]U_aU_b - g_{ab}[L_x\bar{P} + \psi\bar{P}] + [L_xP - L_x\bar{P} + \psi(P - \bar{P})]H_aH_b \quad \ldots (5.5)$$

We utilize the defining expression of tensor R_{ab} to write the value of Lie derivative of the Ricci tensor field $\bar{x}$ in the form

$$L_xR_{ab} = \frac{1}{2}g^{cd}[(L_xg_{cd})_{;ab} - (L_xg_{bd})_{;ca} - (L_xg_{ad})_{;cb} + (L_xg_{ab})_{;cd}]. \quad \ldots (5.6)$$

If the conditions (5.1) are used this equation reduces to

$$L_xR_{ab} = \psi_{;ab} + \frac{1}{2}g_{ab}(\phi\psi), \quad \ldots (5.7)$$

Where $\phi\psi = \psi_{;ab}g^{ab}$.

In the similar way we derive

$$L_xR = L_x(g^{ab}R_{ab}) = 3\phi\psi - \psi R. \quad \ldots (5.8)$$

Thus we find the value of Einstein's tensor $G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}$ as given by

$$L_xG_{ab} = \psi_{;ab} - \frac{1}{2}g_{ab}\phi\psi. \quad \ldots (5.9)$$

Further the Einstein's field equations (3.1) with the use of expressions (5.9) and (5.5) generate the result,

$$\psi_{;ab} - \frac{1}{2}g_{ab}\phi\psi = -k[L_x\bar{\rho} + L_x\bar{P} + \psi(\bar{\rho} + \bar{P})]U_aU_b - g_{ab}[L_x\bar{P} + \psi\bar{P}] + [L_xP - L_x\bar{P} + \psi(P - \bar{P})]H_aH_b. \quad \ldots (5.10)$$

Now we prove that (1) ⇒ (2).

We have, by (1).

$$\psi_{;ab} = 0. \quad \ldots (5.11)$$

This implies that

$$\phi\psi = 0. \quad \ldots (5.12)$$

Hence the left hand side of (5.10) is zero, which gives

$$[L_x\bar{\rho} + L_x\bar{P} + \psi(\bar{\rho} + \bar{P})]U_aU_b - g_{ab}[L_x\bar{P} + \psi\bar{P}] + [L_xP - L_x\bar{P} + \psi(P - \bar{P})]H_aH_b = 0 \quad \ldots (5.13)$$

This equation when transvected with U^aU^b, h^ah^b *and* S^aS^b provides the following required conditions:

$$L_x\bar{\rho} + \psi\bar{\rho} = 0;$$
$$L_x\bar{P} + \psi\bar{P} = 0;$$
$$L_xP + \psi P = 0. \quad \ldots (5.14)$$

This proves that (1) ⇒ (2).

Now, we establish (2) ⇒ (1).

If we use the condition (5.14) in the equation (5.10) we get

$$\psi_{;ab} - \frac{1}{2}g_{ab}\phi\psi = 0. \quad \ldots (5.15)$$

This immediately gives

$$\phi\psi = 0. \quad \ldots (5.16)$$

Consequently, (5.15) yields

$$\psi_{;ab} = 0. \qquad \ldots (5.16')$$

Hence, the proof of the theorem is complete.

Remark 1: This theorem provides the necessary and sufficient dynamical conditions for $\psi_{;ab} = 0$.

Remark 2: For special conformal motions as considered by Herrera e-t al [3] the condition $\psi_{;ab} = 0$ is Self-evident.

Further by utilizing the condition (5.1) –of conformal motion along the arbitrary vector field $\bar{x}$, one can derive conservation law in the form

$$(R^a_b x^b)_{;a} = 0. \qquad \ldots (5.17)$$

For this conservation law we study the following two cases.

Case (1). The field vector$\bar{x}$ has the direction of magnetic field vector H_a i.e. $x_a = hH_a$. ... (5.18)

Now (5.1) and (5.17) yield the result by the use of (3.2)

$$k\left[\frac{\bar{p}}{2} + \frac{P}{2} + \bar{P}\right] = 0. \qquad \ldots (5.19)$$

This implies $\bar{p} + P - 2\bar{P} = 0$, as $\psi \neq 0$. ... (5.20)

Remark 3: It follows from (5.20) that the anisotropic Magnetofluid admitting the group of conformal motions satisfies the equation of state $\bar{p} + P = 2P$

Case (2). The field vector $\bar{x}$ has the direction of S_a i.e.

$$x_a = \alpha S_a \text{ , } \alpha \text{is an scalar.} \qquad \ldots (5.21)$$

With the choice of conservation law (5.17) with the value of R_{ab} given by (3.2) produces $\frac{k}{2}(\bar{p} - P) = 0$,

i.e. $\bar{p} = P, k \neq 0$. ... (5.22)

Remark 4: The equation (5.22) gives the equation of state for the Magnetofluid allowing a group of conformal motion along the vector S_a which is normal to both $U_a$and H^a.

Remark 5: The equations of states as obtained by (5.20) and (5.22) are quite different than the equations of state derived by Herrera et al. [3]. This change is due to the presence of the magnetic field only.

6. Conformal motions along some preferred directions.

$$X_a = \xi U_a. \qquad \ldots (6.1)$$

For this case the condition for conformal motions (5.1) gives

$$\underset{\xi U^a}{\pounds} g_{ab} = \psi g_{ab}. \qquad \ldots (6.2)$$

This will imply

$$\dot{\xi} = \frac{\psi}{2} \qquad \ldots (6.3)$$

And

$$\xi\theta = \frac{3}{2}\psi \qquad \ldots (6.4)$$

Further, the corresponding conservation law provides

$$(R^a_b \xi U^b)_{;a} = 0, \qquad \ldots (6.5)$$

i.e. $$(\bar{p} + P + 2\bar{P})[\xi(\theta - \psi) + \dot{\xi}] = 0 \qquad \ldots (6.6)$$

This when supplemented with the conditions (6.3) and (6.4) gives the result

$$\psi = 0 \text{ when } (\bar{p} + P + 2\bar{P}) \neq 0. \qquad \ldots (6.7)$$

Hence we conclude that the conformal motion along time like direction is not admissible for the anisotropic fluid distribution since in this case the conformal motion degenerates into a group of motions.

Case(3). Conformal motions in the direction parallel to magnetic field vector.
Hence we write

$$x_a = hH_a. \quad \dots (6.8)$$

For this choice the conditions (5.1) imply

$$h_{,a}H_b + h_{,b}H_a + h(H_{a;b} + H_{b;a}) = \psi g_{ab}. \quad \dots (6.9)$$

The contradiction for this supply

$$\psi = 2h_{,a}H^a. \quad \dots (6.10)$$

$$h_{,a}H^a + hH^a{}_{;a} = 2\psi. \quad \dots (6.11)$$

Consequently

$$hH^a{}_{;a} = 3h_{,a}H^a. \quad \dots (6.12)$$

We use this equation (6.12) in equation (3.8) to obtain an intersecting result

$$4h_{,a}H^a + h\dot{u}_a H^a = 0. \quad \dots (6.13)$$

Conclusion: We get from (6.3)

$$h_{,a}H^a = 0 \Leftrightarrow \dot{U}_a H^a = 0. \quad \dots (6.14)$$

This states that in space time of the anisotropic Magnetofluid admitting a group of conformal motions in the direction of magnetic field vector the magnitude of the magnetic field is preserved along the magnetic lines if and only if the four acceleration of the fluid is orthogonal to magnetic lines. Further we have by definition of Lie derivative

$$\frac{L}{h}U_b = h^a[(U_b)_{;a} - (U_a)_{;b}]. \quad \dots (6.15)$$

But we have the expression

$$U_{a;b} = \sigma_{ab} + W_{ab} + \frac{1}{3}\theta h_{ab} + \dot{U}_a U_b \quad \dots (6.16)$$

Due to Greenberg [2] and

$$\frac{L}{h}U_a = \left(\frac{\psi}{2}\right) U_a \quad \dots (6.17)$$

Due to Herrera et al.[3]. Hence by virtue of equations (6.15) – (6.17) we get the results

$$\psi = -2\dot{U}_a h^a, \quad \dots (6.18)$$

And

$$W_{ab}h^b = 0. \quad \dots (6.19)$$

It follows from (6.19) that the magnetic field is normal to the plane of rotation.

Conclusion: For the space time of Magnetofluid admitting a group of conformal motions along the magnetic field vector the magnetic field lines are always perpendicular to the plane of rotation.

7. Concluding remarks

A class of exact solutions of anisotropic Magnetofluid space time admitting conformal motions is found by Surve and Asgekar [5]. These models are described by the line element

$$ds^2 = (mr)^2 dt^2 - (2n\psi^{-1})^2 dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) \quad \dots (7.1)$$

With ψ as arbitrary function of r.

(a) The results (5.20), (6.7), (6.18), (6.19) are consistent with the particular solutions of (7.1).

(b) The deduction (5.22) generates a differential equation the solution of which belongs to the class of solution given by (7.1).

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